Circulant Graphs $G(n, k)$ Are $k$-Hamiltonian When $k = 4$

Shih-Yan Chen$^a$, Tung-Yang Ho$^b$, and Shin-Shin Kao$^{c*}$

$^a,c$ Department of Applied Mathematics
Chung Yuan Christian University,
Chungli City, Taiwan 320, R.O.C.
$^a$yan@blsh.tp.edu.tw, $^c$skao@math.cycu.edu.tw

$^b$Department of Industrial Engineering and Management
Ta Hwa Institute of Technology,
Hsinchu, Taiwan 307, R.O.C.
hoho@thit.edu.tw

Abstract

Let $G$ be a graph. For a positive integer $k$, the $k$-th power $G^k$ of $G$ is the graph having the same vertex set as $G$ such that any two vertices $u$ and $v$ are adjacent in $G^k$ if and only if the distance between $u$ and $v$ in $G$ is at most $k$. A graph $G$ is $k$-hamiltonian if $G - S$ is hamiltonian for any set $S \subset V(G) \cup E(G)$ with $|S| = k$. The graph $G(n, k)$ is the $((k/2) + 1)$-power $C_n^{(k/2)+1}$ of the cycle $C_n$ of order $n$ if $k$ is even, and is the graph obtained from $C_n^{(k+1)/2}$ by adding all or part of the diameters if $k$ is odd. Sung et al.
[1] proved that $G(n, k)$ is $k$-hamiltonian with $k = 2$ and $3$. In this paper, we show that $G(n, k)$ is $k$-hamiltonian for $k = 4$.

Keywords: hamiltonian, hamiltonian connected, circulant graph.

1 Introduction

In this paper, all graphs are undirected and simple. The sets of vertices and edges of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. If $u, v \in V(G)$ and $e = (u, v) \in E(G)$ is an edge between $u$ and $v$, then we say that the vertices $u$ and $v$ are adjacent in $G$, the edge $e$ is incident with $u$ and $v$, and $u$ (or $v$) is an endvertex of $e$. A path $P$ between two vertices $v_0$ and $v_k$ is represented by $P = \langle v_0, v_1, \ldots, v_k \rangle$, where each pair of consecutive vertices are connected by an edge. We also write the path $P = \langle v_0, v_1, \ldots, v_k \rangle$ as $\langle v_0, v_1, \ldots, v_i, Q, v_j, v_{j+1}, \ldots, v_k \rangle$, where $Q$ denotes the path $\langle v_i, v_{i+1}, \ldots, v_j \rangle$. A path in $G$ is called a hamiltonian path of $G$ if it visits every vertex of $G$ exactly once. A cycle is a path of at least three vertices such that the first vertex is the same as the last vertex. A cycle containing all the vertices of a graph $G$ is said to be a hamiltonian cycle of $G$. A graph $G$ containing a hamiltonian cycle is called a hamiltonian graph.
A graph $G$ is $k$-hamiltonian if $G - S$ is hamiltonian for any set $S \subseteq V(G) \cup E(G)$ with $|S| = k$. In particular, a graph $G$ is said to be $k$-vertex hamiltonian (resp. $k$-edge hamiltonian) if $G - S$ is hamiltonian for any set $S \subseteq V(G)$ (resp. $S \subseteq E(G)$) with $|S| = k$.

For a positive integer $k$, the $k$-th power $G^k$ of $G$ is the graph having the same vertex set as $G$ such that any two vertices $u$ and $v$ are adjacent in $G^k$ if and only if the distance between $u$ and $v$ in $G$ is at most $k$. A graph $G$ is a circulant graph with the distance sequence $(d_1, d_2, \ldots, d_k)$ if $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(G) = \{(v_i, v_j) \mid (i - j) \mod n = d_i\}$, for all $1 \leq i \leq k, 0 \leq i, j \leq n - 1$.

The graph $G(n, k)$ is the $(\frac{k}{2} + 1)$-power $C_n^{\frac{k}{2}+1}$ of the cycle $C_n$ of order $n$ if $k$ is even, and is the graph obtained from $C_n^{\frac{k}{2}+1}$ by adding all or part of the diameters if $k$ is odd. More precisely, given two positive integers $n$ and $k$ with $n > 2k$, $V(G(n, k)) = \{v_0, v_1, \ldots, v_{n-1}\}$ and the vertices $v_i'$s are arranged clockwise with ascending order of the indices. If $k$ is even, $G(n, k)$ is defined as a circulant graph with the distance sequence $\{1, 2, \ldots, \frac{k}{2} + 1\}$. If $k$ is odd and $n$ is even, $G(n, k)$ is defined as a circulant graph with the distance sequence $\{1, 2, \ldots, \frac{k+1}{2}, \frac{n}{2}\}$. Otherwise, $G(n, k)$ is not a circulant graph but has edge set $\{(v_i, v_{i+j}) \mid 0 \leq i \leq n - 1$ and $1 \leq j \leq \frac{k+1}{2}\} \cup \{(v_i, v_{i+n+j}) \mid 0 \leq i \leq \frac{n-3}{2}\} \cup \{(v_0, v_{\frac{n}{2}+1}\}$. Some examples of $G(n, k)$ are depicted in Figure 1.

There has been a lot of investigation on the hamiltonicity of $G(n, k)$. For example, M. Paoli, C.K. Wong and W.W. Wong [2, 3] showed that $G(n, k)$ is $k$-vertex hamiltonian and $k$-edge hamiltonian for every $k$. In [1], Sung et al. confirmed the $k$-hamiltonicity of $G(n, k)$ with $k = 2$ and 3. They also conjectured that $G(n, k)$ is $k$-hamiltonian for every $k$. The goal of this paper is to show that $G(n, k)$ is $k$-hamiltonian for $k = 4$.

## 2 Preliminaries

We state some useful results in this section. Throughout this section, we let $P_n = \langle v_0, v_1, \ldots, v_{n-1}\rangle$ be a path of order $n$.

**Theorem 1.** [3] The graph $G(n, k)$ is $k$-vertex hamiltonian and $k$-edge hamiltonian for every $k$ is even.

**Theorem 2.** [2] The graph $G(n, k)$ is $k$-vertex hamiltonian and $k$-edge hamiltonian for every $k$ is odd.

**Theorem 3.** [1] The graphs $G(n, 2)$ and $G(n, 3)$ are 2-hamiltonian and 3-hamiltonian, respectively.

**Theorem 4.** [4] If $G$ is a connected graph, then $G^k$ is $(k - 2)$-edge hamiltonian if $k \geq 3$ and $|V(G)| \geq k + 1$. Therefore, the cube $P_n^3$ of a path $P_n$ is 1-edge hamiltonian if $n \geq 4$. 

![Figure 1: Examples of $G(n, k)$.](image-url)
### Theorem 5.

\[ v_i = 1 \]

for \( i \geq 2. \)

### Lemma 3.

Let \( S \subset \{ v_0, v_{n-1} \} \) be a hamiltonian path with endvertices \( v_0 \) and \( v_{n-1} \). According to Lemma 3 and Theorem 5, it suffices to consider the case when \( n \) is composed of one vertex and one edge. Let \( S = \{ v, e \} \), where \( v \in (V(P^3_n) - \{ v_0, v_{n-1} \}) \) and \( e \in E(P^3_n) \) is not incident to \( v \). Note that \( P^3_{n-1} \) is a subgraph of \( P^3_n - \{ v \} \). Therefore, by Lemma 2, \( P^3_n - \{ v, e \} \) has a hamiltonian path with endvertices \( v_0 \) and \( v_{n-1} \).

### Corollary 1.

Let \( n \geq 6. \) Then \( P^3_n - S \) has a hamiltonian path with endvertices \( v_0 \) and \( v_{n-1} \) for \( S \subset (V(P^3_n) - \{ v_0, v_{n-1} \}) \cup E(P^3_n) \) with \( |S| \leq 2. \)

### Proof.

According to Lemma 3 and Theorem 5, it suffices to consider the case when \( S \) is composed of one vertex and one edge. Let \( S = \{ v, e \} \), where \( v \in (V(P^3_n) - \{ v_0, v_{n-1} \}) \) and \( e \in E(P^3_n) \) is not incident to \( v \). Note that \( P^3_{n-1} \) is a subgraph of \( P^3_n - \{ v \} \). Therefore, by Lemma 2, \( P^3_n - \{ v, e \} \) has a hamiltonian path with endvertices \( v_0 \) and \( v_{n-1} \).

### 3. The 4-Hamiltonicity of \( G(n, 4) \)

#### Theorem 6.

The graph \( G(n, 4) \) is 4-hamiltonian for \( n \geq 9. \)

#### Proof.

Let \( S \subset V(G(n, 4)) \cup E(G(n, 4)) \) with \( |S| = 4. \) We want to prove that there exists a hamiltonian cycle in \( G(n, 4) - S \). According to Theorem 1, \( G(n, 4) - S \) is hamiltonian for \( S \subset V(G(n, 4)) \) or \( S \subset E(G(n, 4)) \). Hence it suffices to consider the remaining three cases.

#### Case 1.

\( S \) is composed of three vertices and one edge. Without loss of generality, we assume that the three removed vertices are...
$v_0, v_i,$ and $v_j$ with $0 < i < j$ and $i \leq j - i \leq n - j$.

Case 1.1. $j \geq 3$. Since $n - 2 \geq 7$, $G(n - 2, 2)$ is a subgraph of $G(n, 4) - \{v_0, v_j\}$. By Theorem 3, $G(n - 2, 2)$ is a 2-hamiltonian graph. Thus $G(n, 4) - S$ is hamiltonian.

Case 1.2. $j = 2$. Note that $i = 1$. The remaining graph after the removal of the vertices $v_0, v_1$, and $v_2$ of $G(n, 4)$ is isomorphic to the graph $P^3_{n-3}$. By Theorem 4, $P^3_{n-3}$ is 1-edge hamiltonian. Thus $G(n, 4) - S$ is hamiltonian.

Case 2. $S$ is composed of two vertices and two edges. Without loss of generality, we assume that $\{v_0, v_1\} \subset S$, where $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

Case 2.1. $i \geq 3$. Note that $G(n - 2, 2)$ is a subgraph of $G(n, 4) - \{v_0, v_1\}$. Since $n - 2 \geq 7$ and by Theorem 3, $G(n - 2, 2)$ is a 2-hamiltonian graph. Thus $G(n, 4) - S$ is hamiltonian.

Case 2.2. $i = 2$. The remaining graph after the removal of $v_0$ and $v_2$ of $G(n, 4)$ is depicted in Figure 2. Let $G'$ be the subgraph of $G$ induced by $\{v_3, v_4, \ldots, v_{n-1}\}$. Obviously, $G'$ is isomorphic to $P^3_{n-3}$.

Case 2.2.1. $|\{(v_3, v_1), (v_1, v_{n-1})\} \cap S| = 0$. By Theorem 5, $G'$ has a hamiltonian path $Q_1$ with endvertices $v_3$ and $v_{n-1}$ after removing any two edges. Thus, $(v_3, Q_1, v_{n-1}, v_1, v_3)$ is the hamiltonian cycle of $G(n, 4) - S$.

Case 2.2.2. $|\{(v_1, v_3), (v_1, v_{n-1})\} \cap S| = 2$. $(v_1, v_4, v_3, v_5, \ldots, v_{n-3}, v_{n-1}, v_n, v_1)$ is the hamiltonian cycle of $G(n, 4) - S$.

Case 2.2.3. $|\{(v_1, v_3), (v_1, v_{n-1})\} \cap S| = 1$. Without loss of generality, we assume that $(v_1, v_{n-1}) \in S$. Suppose that $(v_1, v_{n-2}), (v_{n-1}, v_{n-2}) \cap S = \emptyset$. By Corollary 1, $G' - \{v_{n-2}\}$ has a hamiltonian path $Q_2$ with endvertices $v_3$ and $v_{n-1}$ after removing any one edge. Thus $(v_3, Q_2, v_{n-1}, v_{n-2}, v_1, v_3)$ is the hamiltonian cycle of $G(n, 4) - S$. Suppose that $(v_1, v_{n-2}) \in S$. $P^3_{n-3}$ is the subgraph of $G'$. By Lemma 1, there exists a hamiltonian path $Q_3$ of $G'$ with endvertices $v_3$ and $v_4$. Thus $(v_4, v_1, v_3, Q_3, v_4)$ is the required cycle. Suppose that $(v_{n-1}, v_{n-2}) \in S$, $(v_1, v_3, \ldots, v_{n-4}, v_{n-1}, v_{n-3}, v_{n-2}, v_1)$ is the required cycle.

Case 2.3. $i = 1$. The remaining graph after the removal of $v_0$ and $v_1$ of $G(n, 4)$ is depicted in Figure 3. Suppose that $(v_{n-1}, v_2) \in S$. $G(n, 4) - \{v_0, v_1, (v_{n-1}, v_2)\}$ is isomorphic to $P^3_{n-2}$ which is 1-edge hamiltonian. Hence, the remaining graph after removing any one edge from $G(n, 4) - \{v_0, v_1, (v_{n-1}, v_2)\}$ contains a hamiltonian cycle. Suppose that $(v_{n-1}, v_2) \notin S$. Since the graph $G(n, 4) - \{v_0, v_1, (v_{n-1}, v_2)\}$ is isomorphic to $P^3_{n-2}$, it has a hamiltonian path $Q_1$ with endvertices $v_2$ and $v_{n-1}$ after removing any two edges. Thus,
$(v_2, Q_4, v_{n-1}, v_2)$ is the required cycle.

Figure 4: $G'' = G(n, 4) - \{v_{n-1}\}$.

**Case 3.** $S$ is composed of one vertex and three edges. Without loss of generality, we assume that $v_{n-1} \in S$. The graph $G(n, 4) - \{v_{n-1}\}$ is depicted in Figure 4. Let $G'' = G(n, 4) - \{v_{n-1}\}$. In this case, all the addition and subtraction are carried with modulo $n - 1$. Let $E(G'') = A_1 \cup A_2 \cup A_3$, where $A_i = \{(v_i, v_j) \in E(G')|j = i + l\}$. Let $S_i = S \cap A_i$ for $i = 1, 2, 3$.

Suppose that $S_3 \neq \emptyset$. For any edge $e \in S_3$, $G(n - 1, 2)$ is a subgraph of $G'' - \{e\}$. By Theorem 3, there exists a hamiltonian cycle after removing any two edges. Now, we assume that $S_3 = \emptyset$. We consider the remaining cases in the following:

**Case 3.1.** $|S_2| = 3$. Cycle $(v_0, v_1, \ldots, v_{n-2}, v_0)$ is the required cycle.

**Case 3.2.** $|S_1| = 1$. Note that $|S_2| = 2$. If $(v_{n-2}, v_0) \notin S_1$, without loss of generality, we suppose that $S_1 = \{(v_i, v_{i+1})\}$, where $0 \leq i \leq \lfloor \frac{n-3}{2} \rfloor$. Thus $(v_0, v_1, v_2, v_3, v_{n-2}, v_0)$ is a hamiltonian cycle of $G(n, 4) - S$. Next, we consider the case that $(v_{n-2}, v_0) \in S_1$ in the following two subcases.

**Case 3.2.1.** $\{(v_{n-3}, v_0), (v_{n-2}, v_1)\} \cap S_2 \geq 1$. Note that $G'' - \{(v_{n-2}, v_0), (v_{n-3}, v_0), (v_{n-2}, v_1)\}$ is isomorphic to $P^3_{n-1}$ and $P^3_{n-1}$ is 1-edge hamiltonian. Thus $G(n, 4) - S$ is hamiltonian.

**Case 3.2.2.** $\{|(v_{n-3}, v_0), (v_{n-2}, v_1)\} \cap S_2 = 0$. If $\{|(v_0, v_2), (v_{n-2}, v_0)\} \cap S_2 = 2$, then $(v_0, v_2, v_3, v_2, v_1, \ldots, v_{n-2}, v_1, v_0)$ is the required cycle. Otherwise, $\{|(v_0, v_2), (v_{n-4}, v_{n-2})\} \cap S_2 \leq 1$. Without loss of generality, we assume that $(v_0, v_2) \notin S_2$. Thus $(v_0, v_2, \ldots, v_{n-2}, v_1, v_0)$ is the required cycle.

**Case 3.3.** $|S_1| = 2$. Note that $|S_2| = 1$.

**Case 3.3.1.** The two edges in $S_1$ are adjacent. Assume that $S_1 = \{(v_{i-1}, v_i), (v_i, v_{i+1})\}$, where $0 \leq i \leq \lfloor \frac{n-2}{2} \rfloor$. $(v_{i-1}, v_i, v_{i+1}, v_{i+2}, v_i, v_{i+3}, \ldots, v_{i-1})$ is the required cycle if $\{|(v_{i+1}, v_{i+2}), (v_i, v_{i+1})\} \cap S_2 = 0$; $(v_{i-1}, v_{i+2}, v_i, v_{i+1}, v_j, v_{j-1}, v_{j+1}, \ldots, v_i)$ is the required cycle if otherwise.

**Case 3.3.2.** The two edges in $S_1$ are not adjacent.

**Case 3.3.2.1.** $(v_{n-2}, v_0) \notin S_1$. Without loss of generality, we assume that $S_1 = \{(v_{i-1}, v_i), (v_j, v_{j+1})|1 \leq i < j \leq n - 3\}$. When $j - i = 4$, $(v_{i-1}, v_{i+1}, v_i, v_{j-1}, v_j, v_{j-2}, v_j, v_{j+1}, \ldots, v_{i-1})$ is the required cycle if $\{|(v_{i-1}, v_{i+1}), (v_j, v_{j+1})\} \cap S_2 = 0$; $(v_{i-1}, v_{i+2}, v_i, v_{i+1}, v_j, v_{j-1}, v_{j+1}, \ldots, v_i)$ is the required cycle if otherwise. When $j - i \geq 1$ and $j - i \neq 4$, the corresponding hamiltonian cycles listed in Table 2. Note that when $j - i = 1$, we assume that $1 \leq i \leq \lfloor \frac{n-3}{2} \rfloor$.

**Case 3.3.2.2.** $(v_{n-2}, v_0) \in S_1$. Without loss of generality, we assume that the other edge in $S_1$ is $(v_i, v_{i+1})$, where $1 \leq i \leq \lfloor \frac{n-3}{2} \rfloor$. When $i = 1$, $(v_0, v_2, \ldots, v_{n-2}, v_1, v_0)$ is the required cycle if $\{|(v_{n-2}, v_1), (v_0, v_2)\} \cap S_2 = 0$, and $(v_0, v_1, v_3, v_2, v_1, \ldots, v_{n-4}, v_{n-2}, v_{n-3}, v_0)$ is the required cycle if otherwise. When $2 \leq i \leq \lfloor \frac{n-3}{2} \rfloor$, $(v_0, \ldots, v_i, v_{i+3}, v_{i+2}, v_i, v_{i+4}, \ldots, v_{n-2}, v_{n-3}, v_0)$ is the required cycle if otherwise.
Case 3.4. $|S_1| = 3$.

Case 3.4.1. $|V(G'')|$ is odd. $(v_1, v_3, v_5, \ldots, v_{n-5}, v_{n-3}, v_0, v_2, v_4, \ldots, v_{n-4}, v_{n-2}, v_1)$ is the required cycle.

Case 3.4.2. $|V(G'')|$ is even. Since $n \geq 9$ and $|S_1| = 3$, there exists an index $i$ such that $\{(v_i, v_{i+1}), (v_{i+2}, v_{i+3})\} \cap S_1 = \emptyset$. Thus, $(v_{i+2}, v_{i+4}, v_{i+6}, \ldots, v_{i-4}, v_{i-2}, v_{i+1}, v_{i-1}, v_{i-3}, v_{i-5}, \ldots, v_{i+5}, v_{i+3}, v_{i+2})$ is the required cycle. □

References


